The Expression of Syzygies among Perpetuants by means of Partitions.

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Contents.

This paper is entirely based upon the partition expression of seminvariants; the previous papers on the subject are, in the order of their publication,

The Author. Seminvariants and Symmetric Functions (*A. J. M.*, Vol. 6, p. 131).

Cayley. A Memoir on Seminvariants (*A. J. M.*, Vol. 7, p. 1).

The Author. On Perpetuants (*A. J. M.*, Vol. 7, p. 26).

A second paper on Perpetuants (*A. J. M.*, Vol. 7, p. 259).

A Memoir on Seminvariants (*A. J. M.*, Vol. 8, p. 1).

Hammond. On Perpetuants with Applications to the Theory of Finite Quantics (*A. J. M.*, Vol. 8, p. 104).

§1.

Assume the binary quantic to be

$$(a_0, a_1, a_2, \ldots)(x_0, x_1)^p,$$

wherein it will be observed the variables are written x_0, x_1, and moreover are supposed hypothetically of weights 0 and 1; any coefficient in the quantic has thus the same weight as the combination of variables which it affects.

Write any two covariants of this quantic

$$u \equiv (A_0, A_1, A_2, \ldots)(x_0, x_1)^s,$$
$$v \equiv (B_0, B_1, B_2, \ldots)(x_0, x_1)^\sigma,$$

and represent the differential operation

$$\frac{d^{p+q}}{dx_0^p\, dx_1^q},$$

by $\xi_0^p \xi_1^q$ or by $\eta_0^p \eta_1^q$,

according as it takes effect upon u or upon v; the operation of Ueberschiebung of order $\varkappa$ is then symbolically $(\xi_0\eta_1 - \xi_1\eta_0)^\kappa$,
and noting that

$$\xi_0^p\xi_1^q\eta_0^{p'}\eta_1^{q'} = \frac{s!\,\sigma!}{(s-p-q)!\,(\sigma-p'-q')!}(A_q, A_{q+1}, A_{q+2}, \ldots)(x_0, x_1)^{s-p-q}$$
$$(B_{q'}, B_{q'+1}, B_{q'+2}, \ldots)(x_0, x_1)^{\sigma-p'-q'},$$

we find at once

$$\frac{(s-\varkappa)!\,(\sigma-\varkappa)!}{s!\qquad \sigma!}(\xi_0\eta_1 - \xi_1\eta_0)^\kappa$$
$$= (A_0, A_1, A_2, \ldots)(x_0, x_1)^{s-\kappa}(B_\kappa, B_{\kappa+1}, B_{\kappa+2} \ldots)(x_0, x_1)^{\sigma-\kappa}$$
$$-\varkappa(A_1, A_2, A_3, \ldots)(x_0, x_1)^{s-\kappa}(B_{\kappa-1}, B_\kappa, B_{\kappa+1} \ldots)(x_0, x_1)^{\sigma-\kappa}$$
$$+\frac{\varkappa.\varkappa-1}{2!}(A_2, A_3, A_4, \ldots)(x_0, x_1)^{s-\kappa}(B_{\kappa-2}, B_{\kappa-1}, B_\kappa, \ldots)(x_0, x_1)^{\sigma-\kappa}$$
$$- \ldots;$$

or if we write
$$(A_q, A_{q+1}, A_{q+2}, \ldots)(x_0, x_1)^{s-\kappa} = S_q,$$
$$(B_{q'}, B_{q'+1}, B_{q'+2}, \ldots)(x_0, x_1)^{\sigma-\kappa} = \Sigma_{q'},$$

we may exhibit the result in the symbolic form

$$\frac{(s-\varkappa)!\,(\sigma-\varkappa)!}{s!\qquad \sigma!}(\xi_0\eta_1 - \xi_1\eta_0)^\kappa = (\Sigma - S)^\kappa,$$

wherein $(\Sigma - S)^\kappa$ is to be expanded in the form

$$\Sigma^\kappa S^0 - \varkappa\Sigma^{\kappa-1}S' + \ldots,$$

and then each power changed into a suffix.

Hence, neglecting a numerical multiplier, the source of the covariant

$$(\xi_0\eta_1 - \xi_1\eta_0)^\kappa$$

is in a symbolic form $(A - B)^\kappa$,

or at length

$$A_\kappa B_0 - \varkappa A_{\kappa-1}B_1 + \frac{\varkappa(\varkappa-1)}{2!}A_{\kappa-2}B_2 - \ldots . + (-)^\kappa A_0 B_\kappa .$$

Giving $\varkappa$ the values $0, 1, 2, 3, \ldots ,$

we write down the series of seminvariants:

$$\begin{aligned}&A_0B_0,\\ &A_1B_0 - A_0B_1,\\ &A_2B_0 - 2A_1B_1 + A_0B_2,\\ &A_3B_0 - 3A_2B_1 + 3A_1B_2 - A_0B_3,\\ &\ldots\ldots\ldots\ldots\ldots\\ &\ldots\ldots\ldots\ldots\ldots\\ &\ldots\ldots\ldots\ldots\ldots\end{aligned}$$

each source is derived from the next preceding by the operation of

$$A_1\partial_{A_0} + A_2\partial_{A_1} + \ldots . - B_1\partial_{B_0} - B_2\partial_{B_1} - \ldots . ,$$

and further, each is annihilated by

$$A_0\partial_{A_1} + 2A_1\partial_{A_2} + \ldots . + B_0\partial_{B_1} + 2B_1\partial_{B_2} + \ldots .$$

§2.

Write the reversor

$$\rho a_1\partial_{a_0} + (\rho - 1)\, a_2\partial_{a_1} + (\rho - 2)\, a_3\partial_{a_2} + \ldots . + a_\rho\partial_{a_{\rho-1}}$$

in the form

$$\rho\left\{a_1\partial_{a_0} + \left(1 - \frac{1}{\rho}\right)a_2\partial_{a_1} + \left(1 - \frac{2}{\rho}\right)a_3\partial_{a_2} + \ldots . + \frac{1}{\rho}\, a_\rho\partial_{a_{\rho-1}}\right\}$$

$= \rho\delta$ suppose; then

$$u = A_0x_0^s + \rho\delta A_0x_0^{s-1}x_1 + \frac{1}{2!}\rho^2\delta^2A_0x_0^{s-2}x_1^2 + \ldots .$$

$$= x_0^s e^{\frac{x_1}{x_0}\rho\delta}A_0 \qquad (s \equiv \rho = \infty);$$

but if the quantic be actually supposed of unlimited order, and A_0 be of degree-weight (α, w), then

$$\delta = a_1\partial_{a_0} + a_2\partial_{a_1} + a_3\partial_{a_2} + \ldots . ,$$

$$\rho = \frac{s}{\alpha} \text{ in the limit,}$$

and the asymptotic form of the covariant u is

$$\left(A_0,\ \frac{\delta A_0}{\alpha},\ \frac{\delta^2 A_0}{\alpha^2},\ \ldots\ldots\right)(x_0,\ x_1)^s,$$

or symbolically,

$$x_0^s\left(1+\frac{x_1}{x_0}\,\frac{\delta}{\alpha}\right)^s A_0.$$

The process of Ueberschiebung now gives the series

$$\begin{array}{l} A_0B_0, \\ \beta\delta A_0 . B_0 - \alpha A_0\delta B_0, \\ \beta^2\delta^2 A_0 . B_0 - 2\alpha\beta\delta A_0 . \delta B_0 + \alpha^2 A_0 . \delta^2 B_0, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots \end{array}$$

and in general,

$$\beta^\kappa\delta^\kappa A_0 . B_0 - \kappa\beta^{\kappa-1}\alpha\delta^{\kappa-1}A_0\delta B_0 + \frac{\kappa . \kappa - 1}{2!}\beta^{\kappa-2}\alpha^2\delta^{\kappa-2}A_0\delta^2 B_0 - \ldots\ldots,$$

or with the proviso that $(\delta A_0)^0 = \delta^0 A_0 = A_0$; $\delta^j A_0 = (\delta A_0)^j$, this is, in symbolic form,

$$(\beta\delta A_0 - \alpha\delta B_0)^\kappa.$$

The result just obtained is seen to be identical with that given by M. Perrin, and partially also by M. Maurice d'Ocagne, in recent numbers of the Comptes Rendus of the Academy of Sciences in Paris.*

§3.

Transformation of the Foregoing Results.

Consider the quantic with derived coefficients

$$(a_0,\ 1!\ a_1,\ 2!\ a_2,\ \ldots\ldots)(x_0,\ x_1)^\rho;$$

the sources become transformed into symmetric functions of the roots of the equation

$$a_0x^n - a_1x^{n-1} + a_2x^{n-2} - \ldots\ldots = 0 \quad (n = \infty)$$

in which no root occurs raised to the power unity; (N. B.—These have been elsewhere called non-unitary symmetric functions);

the reversor

$$\rho a_1\partial_{a_0} + (\rho - 1)\,a_2\partial_{a_1} + (\rho - 2)\,a_3\partial_{a_2} + \ldots\ldots$$

becomes

$$\begin{array}{l} \rho\,(a_1\partial_{a_0} + 2a_2\partial_{a_1} + 3a_3\partial_{a_2} + \ldots\ldots) \\ -\ 2\,(a_2\partial_{a_1} + 3a_3\partial_{a_2} + 6a_4\partial_{a_3} + \ldots\ldots), \end{array}$$

* The connection of the result with the process of Ueberschiebung, and the fact that it is merely its expression quâ sources of quantics of unlimited order, was not mentioned by either M. Perrin or M. d'Ocagne.

wherein the numerical coefficients in the last portion of the operator are the successive triangular numbers.

Writing this reversor $\rho\delta_1 - 2\delta_2$,

it is necessary to separately examine δ_1 and δ_2. Put

$$d_\lambda \equiv a_0\partial_{a_\lambda} + a_1\partial_{a_{\lambda+1}} + a_2\partial_{a_{\lambda+2}} + \ldots.,$$

so that

$$\partial_{a_\lambda} = h_0\frac{d_\lambda}{a_0} - h_1\frac{d_{\lambda+1}}{a_0^2} + h_2\frac{d_{\lambda+2}}{a_0^3} - \ldots.,$$

wherein h_w denotes the sum of all the homogeneous products of the roots of the equation, of weight w; thus,

$$h_w = \Sigma(-)^{w+\alpha_1+\alpha_2+\ldots.}\frac{(\alpha_1+\alpha_2+\ldots.)!}{\alpha_1!\ \alpha_2!\ldots.}a_0^{\alpha_0}a_1^{\alpha_1}a_2^{\alpha_2}\ldots.,\qquad (\Sigma s\alpha_s = w);$$

whence

$$\begin{aligned}\delta_1 &= a_1\partial_{a_0} + 2a_2\partial_{a_1} + 3a_3\partial_{a_2} + 4a_4\partial_{a_3} + \ldots.\\
&= a_1\left(h_0\frac{d_0}{a_0} - h_1\frac{d_1}{a_0^2} + h_2\frac{d_2}{a_0^3} - \ldots.\right)\\
&+ 2a_2\left(h_0\frac{d_1}{a_0} - h_1\frac{d_2}{a_0^2} + h_2\frac{d_3}{a_0^3} - \ldots.\right)\\
&+ 3a_3\left(h_0\frac{d_2}{a_0} - h_1\frac{d_3}{a_0^2} + h_2\frac{d_4}{a_0^3} - \ldots.\right)\\
&+ \ldots\ldots\ldots\ldots\ldots\ldots\\
&= \frac{a_1h_0}{a_0}d_0 - \frac{a_1h_1 - 2a_2a_0h_0}{a_0^2}d_1 + \frac{a_1h_2 - 2a_2a_0h_1 + 3a_3a_0^2h_0}{a_0^3}d_2 - \ldots.\\
&= \sum_{\lambda=0}^{\lambda=\infty}(-)^\lambda\frac{a_1h_\lambda - 2a_2a_0h_{\lambda-1} + 3a_3a_0^2h_{\lambda-2} - \ldots. + (-)^{s+1}Sa_sa_0^{s-1}h_{\lambda-s+1} + \ldots.}{a_0^{\lambda+1}}d_\lambda\\
&= \sum_{\lambda=0}^{\lambda=\infty}(-)^\lambda S_{\lambda+1}d_\lambda \text{ by a known formula.}\end{aligned}$$

S_r denoting the sum of the r^{th} powers of the roots.

$$\therefore \delta_1 = S_1d_0 - S_2d_1 + S_3d_2 - \ldots.$$

Similarly

$$\begin{aligned}\delta_2 &= a_2\partial_{a_1} + 3a_3\partial_{a_2} + 6a_4\partial_{a_3} + \ldots.\\
&= a_2\left(h_0\frac{d_1}{a_0} - h_1\frac{d_2}{a_0^2} + h_2\frac{d_3}{a_0^3} - \ldots.\right)\\
&+ 3a_3\left(h_0\frac{d_2}{a_0} - h_1\frac{d_3}{a_0^2} + h_2\frac{d_4}{a_0^3} - \ldots.\right)\\
&+ 6a_4\left(h_0\frac{d_3}{a_0} - h_1\frac{d_4}{a_0^2} + h_2\frac{d_5}{a_0^3} - \ldots.\right)\\
&+ \ldots.\end{aligned}$$

$$= \frac{a_2h_0}{a_0}d_1 - \frac{a_2h_1 - 3a_3a_0h_0}{a_0^2}d_2 + \frac{a_2h_2 - 3a_3a_0h_1 + 6a_4a_0^2h_0}{a_0^3}d_3 - \dots .$$

$$= \sum_{\lambda=1}^{\lambda=\infty}(-)^{\lambda+1}a_0^{-\lambda}$$

$$\left\{a_2h_{\lambda-1} - 3a_3a_0h_{\lambda-2} + 6a_4a_0^2h_{\lambda-3} + \dots + (-)^{\lambda+1}\frac{1}{2}\lambda(\lambda+1)a_{\lambda+1}a_0^{\lambda-1}h_0\right\}d_\lambda$$

$$= \sum_{\lambda=1}^{\lambda=\infty}(-)^{\lambda+1}S_{\lambda+1}^2d_\lambda \text{ by a known formula.}$$

$S_{\lambda+1}^2$ denoting the sum of all those symmetric functions of weight $\lambda+1$ which involve two and only two roots in their expression; thus,

$$S_{\lambda+1}^2 = \Sigma\Sigma\alpha^p\beta^q *$$

where $$p+q=\lambda+1,$$
$$p>0,\ q>0.$$

We thus have

$$\delta_2 = S_2^2d_1 - S_3^2d_2 + S_4^2d_3 - \dots ,$$

and $$\rho\delta_1 - 2\delta_2 = \rho S_1d_0 - (\rho S_2 + 2S_2^2)d_1 + (\rho S_3 + 2S_3^2)d_2 - \dots ;$$

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* The construction of the analogous function

$$S_w^r \equiv \Sigma_r\alpha_1^{p_1}\alpha_2^{p_2}\dots\alpha_r^{p_r}, \qquad \begin{pmatrix}\Sigma p = w \\ p_1, p_2, p_3, \dots p_r > 0\end{pmatrix}$$

when expressed in terms of the literal coefficients may be exhibited as follows. Let

$$a_0^{\mu_0}a_1^{\mu_1}\dots a_s^{\mu_s}, \dots a_t^{\mu_t}$$

be any product of coefficients of weight w; the corresponding partition of w consists of μ_1 ones, μ_2 twos and so on, and is represented by

$$(1^{\mu_1}2^{\mu_2}\dots s^{\mu_s}\dots t^{\mu_t});$$

we form the function

$$\sum_{s=1}^{s=t}\mu_s(1+x)^s = \sum_{s=1}^{s=t}\mu_s + \sum_{s=1}^{s=t}\mu_s sx + \sum_{s=1}^{s=t}\mu_s\frac{s.s-1}{2!}x^2 + \dots + \sum_{s=1}^{s=t}\mu_s\frac{s!}{r!\,(s-r)!}x^r + \dots$$
$$= W_0 + W_1x + W_2x^2 + \dots + W_rx^r + \dots ,$$

and then $$S_w^r = \Sigma(-)^{W_0+W_1+r+1}\frac{(W_0-1)!\ W_r}{\mu_1!\ \mu_2!\dots\mu_t!}a_0^{\mu_0}a_1^{\mu_1}\dots a_t^{\mu_t}.$$

As a verification, observe that

$$S_w^1 + S_w^2 + \dots + S_w^w = \Sigma\frac{(-)^{W_0+W_1}}{\mu_1!\ \mu_2!\dots\mu_t!}(W_0-1)!\,\{W_1 - W_2 + W_3 - \dots + (-)^{w+1}W_w\}\,a_0^{\mu_0}a_1^{\mu_1}\dots a_t^{\mu_t}$$
$$= \Sigma\frac{(-)^{W_0+W_1}W_0!}{\mu_1!\ \mu_2!\dots\mu_t!}a_0^{\mu_0}a_1^{\mu_1}\dots a_t^{\mu_t}$$
$$= h_w \text{ as should be the case.}$$

δ_1 and δ_2 are each separately reversors; expressed by means of partitions, we find
$$\delta_1 = (1)\,d_0 - (2)\,d_1 + (3)\,d_2 - (4)\,d_3 + \ldots .$$
$$\delta_2 = (1^2)\,d_1 - (21)\,d_2 + \{(31)+(2^2)\}\,d_3 - \{(41)+(32)\}\,d_4 + \ldots .$$

§4.

Operation of the Reversors upon a source which is symbolically expressed by means of Partitions.

It is well known that any source may be expressed symbolically by an aggregate of partitions of its weight, the parts of each partition being drawn from the natural numbers to w inclusive, unity alone being excluded; and further, that the highest symbolic number occurring in the aggregate represents the degree of the source in the coefficients; the operation of d_λ upon any such partition is seen by writing
$$d_\lambda \equiv \Sigma\,(-)^{W_0+W_1}\,\frac{(W_0-1)!\;W_1}{\mu_1!\;\mu_2!\ldots.\mu_t!}\,D_1^{\mu_1}D_2^{\mu_2}\ldots.D_t^{\mu_t}, \qquad \begin{pmatrix} W_0 = \Sigma\mu \\ W_1 = \Sigma s\mu_s \end{pmatrix}$$
where
$$D_\kappa = \frac{1}{\varkappa!}\,(a_0\partial_{a_1} + a_1\partial_{a_2} + a_2\partial_{a_3} + \ldots .)^\kappa,$$
and remarking Hammond's theorem that D_κ operating upon a partition has the effect of striking out therefrom one symbolic number $\varkappa$ when such is possible, and an annihilating effect in every other case. It is hence manifest that δ_1 introduces, by the operation of its member
$$(1)\,d_0,$$
symmetric functions whose partitions contain at most one unit; also in the operation of δ_2, (1^2) will not appear as a multiplier, since in the present case $d_1 \equiv D_1 \equiv 0$, and $S_\lambda^2\,(\lambda > 2)$ is expressed by employing merely a single unit; whence δ_2 also introduces partitions which contain at most one unit, and so also does any linear function of δ_1 and δ_2.

This is obvious, too, from the consideration that any coefficient is derivable from the next succeeding one by the operation of
$$d_1,$$
the effect of which is to obliterate a non-unitary partition and to take away one part unity from all unitary partitions; we now see by a process of induction that a covariant coefficient, s removes from its source, is expressible symbolically by means of partitions, no one of which contains more than s parts equal to unity.

§5.

General Form of a Covariant.

Denote by
$$\theta_s$$
any assemblage of non-unitary positive integers, the sum of which is $w+s$, and of which no number exceeds θ in magnitude; any two covariants then assume the forms

$$(\theta_0)\,x_0^s + s\,\{(\theta_0 1)+(\theta_1)\}\,x_0^{s-1}x_1 + s.s-1\,\{(\theta_0 1^2)+(\theta_1 1)+(\theta_2)\}\,x_0^{s-2}x_1^2 + \ldots\ldots,$$
$$(\phi_0)\,x_0^\sigma + \sigma\{(\phi_0 1)+(\phi_1)\}x_0^{\sigma-1}x_1 + \sigma.\sigma-1\,\{(\phi_0 1^2)+(\phi_1 1)+(\phi_2)\}\,x_0^{\sigma-2}x_1^2 + \ldots\ldots,$$

to which, applying Ueberschiebung of order $\varkappa$, there results the source

$$\begin{aligned}&(\theta_0)\,\{(\phi_0 1^\kappa)+(\phi_1 1^{\kappa-1})+\ldots\ldots+(\phi_\kappa)\}\\ &-\{(\theta_0 1)+(\theta_1)\}\{(\phi_0 1^{\kappa-1})+(\phi_1 1^{\kappa-2})+\ldots\ldots+(\phi_{\kappa-1})\}\\ &+\ldots\ldots\\ &+(-)^\kappa\{(\theta_0 1^\kappa)+(\theta_1 1^{\kappa-1})+\ldots\ldots+(\theta_\kappa)\}(\phi_0),\end{aligned}$$

an expression which consists of $\frac{1}{6}(\varkappa+1)(\varkappa+2)(\varkappa+3)$ binary products, and which may be put into the form:

$$\begin{aligned}&\sum_{j=0}^{j=\kappa}\{(\theta_0)(\phi_j 1^{\kappa-j})-(\theta_0 1)(\phi_j 1^{\kappa-j-1})+(\theta_0 1^2)(\phi_j 1^{\kappa-j-2})-\ldots\ldots+(-)^{\kappa-j}(\theta_0 1^{\kappa-j})(\phi_j)\}\\ &-\sum_{j=0}^{j=\kappa-1}\{(\theta_1)(\phi_j 1^{\kappa-j-1})-(\theta_1 1)(\phi_j 1^{\kappa-j-2})+(\theta_1 1^2)(\phi_j 1^{\kappa-j-3})\\ &\qquad\qquad-\ldots\ldots+(-)^{\kappa-j-1}(\theta_1 1^{\kappa-j-1})(\phi_j)\}\\ &+\ldots\ldots\\ &+(-)^\kappa(\theta_\kappa)(\phi_0);\end{aligned}$$

or as the double sum

$$\begin{aligned}&\sum_{i=0}^{i=\kappa}\sum_{j=0}^{j=\kappa-i}(-)^i\{(\theta_i)(\phi_j 1^{\kappa-i-j})-(\theta_i 1)(\phi_j 1^{\kappa-i-j-1})\\ &\qquad+\ldots\ldots+(-)^{\kappa-i-j+1}(\theta_i 1^{\kappa-i-j-1})(\phi_j 1)+(-)^{\kappa-i-j}(\theta_i 1^{\kappa-i-j})(\phi_j)\};\end{aligned}$$

which constitutes the partition expression of the source of the covariant which is derived from Ueberschiebung of order $\varkappa$ applied to the two given covariants.

The source clearly consists of $\frac{(\varkappa+1)(\varkappa+2)}{2}$ $(=1+2+\ldots\ldots+\varkappa+1)$ distinct parts, or members, a number which, in the general case here considered, cannot be exceeded, and which usually will not be reached.

Theorem. "Each constituent member of a source is itself a seminvariant."

For operating upon any expression of the form

$$(\theta)(\phi 1^t)-(\theta 1)(\phi 1^{t-1})+\ldots\ldots+(-)^{t-1}(\theta 1^{t-1})(\phi 1)+(-)^t(\theta 1^t)(\phi)$$

with the annihilator $\qquad d_1 \equiv D_1$ (*ante*),

we find

$$(\theta)(\phi 1^{t-1})-(\theta)(\phi 1^{t-1})+(\theta 1)(\phi 1^{t-2})-(\theta 1)(\phi 1^{t-2})$$
$$+\ldots\ldots+(-)^{t-1}(\theta 1^{t-1})(\phi)+(-)^t(\theta 1^{t-1})(\phi),$$

which vanishes, the terms destroying each other in pairs.

Each constituent member is therefore a non-unitary symmetric function, and the process of Ueberschiebung is seen to produce sources, each of which is an aggregate of symmetrical members of the same type.*

§6.

Incorporation of Two Sources.

I call, provisionally, the expression

$$(\theta_0)(\phi_0 1^\kappa)-(\theta_0 1)(\phi_0 1^{\kappa-1})+\ldots\ldots+(-)^\kappa(\theta_0 1^\kappa)(\phi_0)$$

the incorporation of (θ_0) and (ϕ_0) of order κ, or say the κ^{th} incorporation of (θ_0) and (ϕ_0). I denote this by $\qquad |(\theta_0)(\phi_0)|^\kappa$.

From the previous section it will be seen that the κ^{th} Ueberschiebung of (θ_0) and (ϕ_0) is composed in general of

1	incorporation	of order	κ,
2	"	"	$\kappa-1$,
3	"	"	$\kappa-2$,
. . .	. . .	. . .	. . .
$\kappa+1$	"	"	0;

* This theorem may be extended in the following manner. Forming the expression

$$(\theta 1^{s-1})(\phi 1^t)-s(\theta 1^s)(\phi 1^{t-1})+\frac{s.s+1}{2!}(\theta 1^{s+1})(\phi 1^{t-2})-\ldots\ldots+(-)^t\frac{(s+t-1)!}{(s-1)!\,t!}(\theta 1^{s+t-1})(\phi),$$

and operating with D_1 we find

$$(\theta 1^{s-2})(\phi 1^t)-(s-1)(\theta 1^{s-1})(\phi 1^{t-1})+\frac{s-1.s}{2!}(\theta 1^s)(\phi 1^{t-2})-\ldots\ldots+(-)^t\frac{(s+t-2)!}{(s-2)!\,t!}(\theta 1^{s+t-2})(\phi),$$

which is derived from the former by writing $s-1$ for s; whence, assuming it to have been shown that the latter expression is expressible by partitions each of which contains at most $s-2$ units, it has been proved that the former is expressible by partitions each of which contains at most $s-1$ units; it has been proved to be true in the latter case when $s=2$, and hence by induction the theorem is true for any value of s.

thus a $\varkappa^{\text{th}}$ Ueberschiebung is composed of a $\varkappa^{\text{th}}$ incorporation of the same two sources, together with other incorporations of lower order; conversely, a $\varkappa^{\text{th}}$ incorporation is expressible by means of a $\varkappa^{\text{th}}$ Ueberschiebung of the same two sources together with incorporations of lower order; hence it is an immediate inference that a $\varkappa^{\text{th}}$ incorporation is expressible by means of a $\varkappa^{\text{th}}$ Ueberschiebung of the same two sources together with Ueberschiebungen of lower order. The forms which here present themselves do not in general belong to the original quantic, and for this reason, in what follows, the quantic will be supposed of indefinite order unless the contrary be expressly stated.

If in the $\varkappa^{\text{th}}$ incorporation of (θ_0) and (ϕ_0) we write

$$(\theta_0 1^i) = a_i,$$
$$(\phi_0 1^i) = b_i,$$

it assumes the form

$$a_0 b_\kappa - a_1 b_{\kappa-1} + a_2 b_{\kappa-2} - \ldots + (-)^\kappa a_\kappa b_0,$$

which is the source of a covariant of the system of two quantics

$$(a_0,\ 1!\ a_1,\ 2!\ a_2,\ \ldots)(x_0,\ x_1)^{\rho_1},$$
$$(b_0,\ 1!\ b_1,\ 2!\ b_2,\ \ldots)(x_0,\ x_1)^{\rho_2},$$

from the simple fact that $\quad d_1(\theta_0 1^i) = (\theta_0 1^{i-1})$,
it is evident, *à priori*, that if

$$f(a_0,\ a_1,\ a_2,\ \ldots\ b_0,\ b_1,\ b_2,\ \ldots\ c_0,\ c_1,\ c_2,\ \ldots)$$

be the source of any covariant of the system of quantics

$$(a_0,\ 1!\ a_1,\ 2!\ a_2,\ \ldots)(x_0,\ x_1)^{\rho_1},$$
$$(b_0,\ 1!\ b_1,\ 2!\ b_2,\ \ldots)(x_0,\ x_1)^{\rho_2},$$
$$(c_0,\ 1!\ c_1,\ 2!\ c_2,\ \ldots)(x_0,\ x_1)^{\rho_3},$$
$$\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots$$

the expression

$$f\{(\theta_0),\ (\theta_0 1),\ (\theta_0 1^2)\ \ldots\ (\phi_0),\ (\phi_0 1),\ (\phi_0 1^2)\ \ldots\ (\psi_0),\ (\psi_0 1),\ (\psi_0 1^2)\ \ldots\}$$

must be a seminvariant; the partitions refer as usual to the same quantic of infinite order.

The process of incorporation is thus a very particular case of this generalization.

§7.

Binomial Syzygies.

If θ_0, ϕ_0, ψ_0 retain their original significations, we may write down the absolute identity

$$\begin{vmatrix} (\theta_0)\ , & (\phi_0)\ , & (\psi_0) \\ (\theta_0)\ , & (\phi_0)\ , & (\psi_0) \\ (\theta_0 1), & (\phi_0 1), & (\psi_0 1) \end{vmatrix} = 0,$$

that is,

$$\begin{aligned} &(\theta_0)\{(\phi_0)(\psi_0 1) - (\phi_0 1)(\psi_0)\} \\ &+ (\phi_0)\{(\psi_0)(\theta_0 1) - (\psi_0 1)(\theta_0)\} \\ &+ (\psi_0)\{(\theta_0)(\phi_0 1) - (\theta_0 1)(\phi_0)\} = 0; \end{aligned}$$

this identity manifestly represents a syzygy between the three compound seminvariants

$$\begin{aligned} &(\theta_0)\{(\phi_0)(\psi_0 1) - (\phi_0 1)(\psi_0)\}, \\ &(\phi_0)\{(\psi_0)(\theta_0 1) - (\psi_0 1)(\theta_0)\}, \\ &(\psi_0)\{(\theta_0)(\phi_0 1) - (\theta_0 1)(\phi_0)\}; \end{aligned}$$

an application to the theory of perpetuants may be at once shown, for write

$$(\theta_0, \phi_0, \psi_0) = (2^\kappa, 2^\lambda, 2^\mu),$$

and then

$$\begin{aligned} &(2^\kappa)\{(2^\lambda)(2^\mu 1) - (2^\lambda 1)(2^\mu)\} \\ &+ (2^\lambda)\{(2^\mu)(2^\kappa 1) - (2^\mu 1)(2^\kappa)\} \\ &+ (2^\mu)\{(2^\kappa)(2^\lambda 1) - (2^\kappa 1)(2^\lambda)\} = 0; \end{aligned}$$

the expression

$$(2^\lambda)(2^\mu 1) - (2^\lambda 1)(2^\mu),$$

as well as the other two expressions obtained from it by the cyclical substitution $(\kappa\lambda\mu)$, consists, when developed in a series of monomial symmetric functions, of quartic and cubic perpetuants; as a consequence, the identity represents a sextic syzygy between binary products of quartic and quadric perpetuants.

Putting now $\mu = 0$, there results the identity

$$\begin{aligned} &(2^\kappa)\{(2^\lambda)(1) - (2^\lambda 1)\} \\ &+ (2^\lambda)\{(2^\kappa 1) - (1)(2^\kappa)\} \\ &+ \quad \{(2^\kappa)(2^\lambda 1) - (2^\kappa 1)(2^\lambda)\} = 0; \end{aligned}$$

herein

$$\begin{aligned} &(2^\lambda)(1) - (2^\lambda 1) = (3 2^{\lambda-1}), \\ &(2^\kappa 1) - (1)(2^\kappa) = -(3 2^{\kappa-1}); \end{aligned}$$

whence

$$(3 2^{\lambda-1})(2^\kappa) - (3 2^{\kappa-1})(2^\lambda) = (2^\lambda)(2^\kappa 1) - (2^\lambda 1)(2^\kappa);$$

remarking that the right-hand side of this identity consists wholly of quartic and cubic perpetuants, we see that we have here exhibited a quintic syzygy between the binary products on the left-hand side.

It is easy to see otherwise that we have the congruence

$$(32^{\lambda-1})(2^{\kappa})-(32^{\kappa-1})(2^{\lambda})\equiv 0 \bmod a,$$

for

$$(32^{\lambda-1})(2^{\kappa})-(32^{\kappa-1})(2^{\lambda})$$

certainly contains when developed no symbolic numbers greater than 5; but also

$$D_5\{(32^{\lambda-1})(2^{\kappa})-(32^{\kappa-1})(2^{\lambda})\}=(2^{\lambda-1})(2^{\kappa-1})-(2^{\kappa-1})(2^{\lambda-1})=0;$$

so that the symbolic number 5 does not occur; and also

$$D_4\{(32^{\lambda-1})(2^{\kappa})-(32^{\kappa-1})(2^{\lambda})\}=(32^{\lambda-2})(2^{\kappa-1})-(32^{\kappa-2})(2^{\lambda-1}),$$

which does not vanish; the symbolic number 4 does occur therefore in the development, and this establishes that the expression

$$(32^{\lambda-1})(2^{\kappa})-(32^{\kappa-1})(2^{\lambda})$$

contains as a factor the seminvariant a raised to the first and to no higher power. The above written congruence is thus verified.

Moreover, the syzygy that has been obtained, includes, for suitable values of κ and λ, every quintic syzygy in the theory of perpetuants.

To establish this it is merely necessary to note the effect of operating upon the left-hand side with D_5; it is then seen that to a syzygy, of weight w, corresponds invariably either a quadric or a quartic compound form which is not a perfect square; since the generating function for these forms is

$$\frac{x^2}{1-x^2}+\frac{x^6}{1-x^2.1-x^4}=\frac{x^2}{1-x^2.1-x^4},$$

it at once follows that the generating function for independent syzygies is

$$x^5.\frac{x^2}{1-x^2.1-x^4}=\frac{x^7}{1-x^2.1-x^4},$$

which we, otherwise, know to represent the total number (cf. Hammond, *Amer. Journ. Math.*, Vol. 4, p. 218).

Reverting to the above general sextic syzygy, let us suppose

$$\kappa \geqq \lambda > \mu,$$

and then operate upon the identity with

$$D_6^{\mu};$$

we thus obtain

$$\begin{aligned}&(2^{\kappa-\mu})\{(2^{\lambda-\mu})(1)-(2^{\lambda-\mu}1)\}\\ &+(2^{\lambda-\mu})\{(2^{\kappa-\mu}1)-(1)(2^{\kappa-\mu})\}\\ &+\{(2^{\kappa-\mu})(2^{\lambda-\mu}1)-(2^{\kappa-\mu}1)(2^{\lambda-\mu})\}=0,\end{aligned}$$

which is equivalent to the result obtained by simply giving μ a zero value in the original identity. In Professor Cayley's language, the effect of the operation of D_6 is, in the present instance, to decapitate the identity. We have in fact here performed μ successive operations of decapitation, and we have thereby arrived at the most general quintic syzygy; it hence follows that the sextic syzygies we are now discussing are those derivable by an infinite number of processes of (4.2) capitation of the quintic syzygies. Their generating is manifestly

$$\frac{x^7}{(1-x^2)(1-x^4)} \cdot \frac{x^6}{1-x^6} = \frac{x^{13}}{(1-x^2)(1-x^4)(1-x^6)};$$

altogether, syzygies, enumerated by means of the generating function

$$\frac{x^7}{(1-x^2)(1-x^4)(1-x^6)}$$

are exhibited in a crystalline form by means of a single absolute identity.

As another example take

$$(\theta_0, \phi_0, \psi_0) = (2^\kappa, a_0, 3^\lambda 2^\mu),$$

and then

$$\begin{aligned}&(2^\kappa)\{(3^\lambda 2^\mu 1) - (1)(3^\kappa 2^\lambda)\}\\ &+ (3^\lambda 2^\mu)\{(2^\kappa)(1) - (2^\kappa 1)\}\\ &+ \{(3^\lambda 2^\mu)(2^\kappa 1) - (3^\lambda 2^\mu 1)(2^\kappa)\} = 0,\end{aligned}$$

an identity which will be shown to represent every binomial sextic syzygy in the theory of perpetuants. It may in fact be written

$$(43^{\lambda-1}2^\mu)(2^\kappa) - (3^\lambda 2^\mu)(3\,2^{\kappa-1}) = \{(3^\lambda 2^\mu)(2^\kappa 1) - (3^\lambda 2^\mu 1)(2^\kappa)\} - (\lambda+1)(3^{\lambda+1}2^{\mu-1})(2^\kappa),$$

and hence a reference to my first paper on perpetuants in the *American Journal of Mathematics* (Vol. 7, p. 26) will show that the statement is true.

The fact is, that all simple syzygies may be exhibited and enumerated in a precisely similar manner, but it is not appropriate to further multiply examples of the method. The discussion of the capitation syzygies in general is, from their inherent nature, a far more difficult matter, and all that will be done here subsequently will be to indicate to what extent progress has been made.

§8.

Particular Forms of Incorporation.

Consider the form

$$|(2^j), a_0|^m \equiv (2^j)(1^m) - (2^j 1)(1^{m-1}) + (2^j 1^2)(1^{m-2}) - \ldots + (-)^m (2^j 1^m),$$

and the result of expanding it in a series of monomials. Since

$$D_3 \,|\,(2^j),\, a_0\,|^m = |\,(2^{j-1}),\, a_0\,|^{m-1},$$

it appears that having expanded $|\,(2^{j-1}),\, a_0\,|^{m-1}$, we have merely to capitate each partition therein presenting itself with the symbolic number 3, in order at once to obtain all those partitions in the expansion of

$$|\,(2^j),\, a_0\,|^m$$

in which a symbolic number 3 appears; thus, since

$$|\,(2^{j-1}),\, a_0\,|^1 = (3\,2^{j-2}),$$

we at once find

$$|\,(2^j),\, a_0\,|^2 = (3^2 2^{j-2}) + \ldots\,;$$

hence, to obtain the general formula, it is only necessary to find that term in the expansion which is made up wholly of twos. Such a term only exists for an even value of m, and the only product of partitions in

$$|\,(2^j),\, a_0\,|^m$$

which can possibly give rise to it is

$$(-)^{\frac{1}{2}m}(2^j 1^{\frac{1}{2}m})(1^{\frac{1}{2}m}),$$

the expansion of which includes the term

$$(-)^{\frac{1}{2}m}\frac{\left(j+\frac{1}{2}m\right)!}{\frac{m}{2}!\; j!}\,(2^{j+\frac{1}{2}m})$$

(see Cayley, *A. J. of M.*, Vol. 7, p. 1); hence, by an easy process, we reach the formula

$$|\,(2^j),\, a_0\,|^m = \sum_{s=0}^{s \,\overline{\gtrless}\, \frac{1}{2}m} (-)^s \frac{(j-m+3s)!}{s!\,(j-m+2s)!}\,(3^{m-2s}2^{j-m+3s}),$$

which exhibits the m^{th} incorporation of the general quadric form (2^j) with the simplest seminvariant a_0. Conversely, we can express any single partition cubic form in terms of incorporation, s, thus

$$(3\,2^\kappa) = |\,(2^{\kappa+1}),\, a_0\,|^1,$$
$$(3^2 2^\kappa) = |\,(2^{\kappa+2}),\, a_0\,|^2 + (\kappa+3)\,|\,(2^{\kappa+3}),\, a_0\,|^0,$$

and in general

$$(3^p 2^\kappa) = \sum_{s=0}^{s \,\overline{\gtrless}\, \frac{1}{2}p} \frac{(\kappa+3s).(\kappa+s-1)!}{\kappa!\; s!}\,|\,(2^{\kappa+p+s}),\, a_0\,|^{p-2s}.$$

Consider now the incorporation $|(3^\kappa 2^\lambda),\ a_0|^m$. This form is a quartic seminvariant; for reasons similar to those above given in the previous case, and which are consequent upon the relation

$$D_4|(3^\kappa 2^\lambda),\ a_0|^m = |(3^{\kappa-1}2^\lambda),\ a_0|^{m-1}.$$

We need only calculate those terms in the expansion which do not contain a symbolic number 4. The product $(-)^s(3^\kappa 2^\lambda 1^s)(1^{m-s})$ which occurs in $|(3^\kappa 2^\lambda),\ a_0|^m$, when multiplied out, only produces one such partition, and that one is

$$(-)^s(3^{\kappa+m-2s}2^{\lambda-m+3s});$$

the numerical coefficient attached to it is (vide Cayley loc. cit.)

$$\frac{(\kappa+m-2s)!\ (\lambda-m+3s)!}{\kappa!\ s!\ (m-2s)!\ (\lambda-m+2s)!},$$

hence the following formula:

$$|(3^\kappa 2^\lambda),\ a_0|^m =$$
$$\sum_{p=0}^{p=m}\ \sum_{s=0}^{s\overline{\gtrless}\frac{1}{2}(m-p)} \frac{(-)^s(\kappa+m-2p-2s)!\ (\lambda-m+p+3s)!}{(\kappa-p)!\ s!\ (m-p-2s)!\ (\lambda-m+p+2s)!}(4^p 3^{\kappa+m-2p-2s}2^{\lambda-m+p+3s}).$$

The incorporation of two quadric forms may be expressed similarly. If we wish to incorporate the seminvariant

$$|(2^\kappa),\ a_0|^m \equiv (2^\kappa)(1^m) - (2^\kappa 1)(1^{m-1}) + \ldots\ldots + (-)^m(2^\kappa 1^m)$$

with a_0, we may proceed in more than one way, but only one really distinct result is obtainable for each order of incorporation. Remembering that the expression

$$(2^\kappa 1^s)(1^m) - (s+1)(2^\kappa 1^{s+1})(1^{m-1}) + \frac{(s+1)(s+2)}{2!}(2^\kappa 1^{s+2})(1^{m-2})$$
$$- \ldots\ldots + (-)^m\frac{(s+m)!}{s!\ m!}(2^\kappa 1^{s+m})$$

consists, when expanded, of partitions containing at most s units, we may form the incorporation of order t:

$$\{(2^\kappa)(1^m) - (2^\kappa 1)(1^{m-1}) \quad + \ldots\ldots + (-)^m(2^\kappa 1^m)\}(1^t)$$
$$-\{(2^\kappa 1)(1^m) - 2(2^\kappa 1^2)(1^{m-1}) + \ldots\ldots + (-)^m m(2^\kappa 1^{m+1})\}(1^{t-1})$$
$$+\{(2^\kappa 1^2)(1^m) - 3(2^\kappa 1^3)(1^{m-1}) + \ldots\ldots + (-)^m\frac{m.m+1}{2!}(2^\kappa 1^{m+2})\}(1^{t-2})$$
$$-\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$+(-)^t\{(2^\kappa 1^t)(1^m) - (t+1)(2^\kappa 1^{t+1})(1^{m-1}) + \ldots\ldots + (-)^m\frac{(m+t-1)!}{t!\ (m-1)!}(2^\kappa 1^{m+t})\},$$

which is

$$(2^\kappa)(1^m)(1^t)-(2^\kappa 1)D_1\{(1^m)(1^t)\}+(2^\kappa 1^2)D_1^2\{(1^m)(1^t)\}$$
$$-\ldots\ldots+(-)^{m+t}(2^\kappa 1^{m+t})D_1^{m+t}\{(1^m)(1^t)\},$$

as may be seen by making a rearrangement of terms. Now

$$(1^m)(1^t)=(2^t 1^{m-t})+(m-t+2)(2^{t-1}1^{m-t+2})$$
$$+\frac{(m-t+3)(m-t+4)}{2!}(2^{t-2}1^{m-t+4})+\ldots\ldots$$

$$\therefore D_1^s\{(1^m)(1^t)\}=(2^t 1^{m-t-s})+(m-t+2)(2^{t-1}1^{m-t-s+2})$$
$$+\frac{(m-t+3)(m-t+4)}{2!}(2^{t-2}1^{m-t-s+4})+\ldots\ldots,$$

whence the incorporation of $|(2^\kappa),\, a_0|^m$ with a_0 may be written:

$$(2^\kappa)(2^t 1^{m-t})-(2^\kappa 1)(2^t 1^{m-t-1})+(2^\kappa 1^2)(2^t 1^{m-t-2})-\ldots\ldots+(-)^{m-t}(2^\kappa 1^{m-t})(2^t)$$
$$+(m-t+2)\{(2^\kappa)(2^{t-1}1^{m-t+2})-(2^\kappa 1)(2^{t-1}1^{m-t+1})+\ldots\ldots+(-)^{m-t}(2^\kappa 1^{m-t+2})(2^{t-1})\}$$
$$+\frac{(m-t+4)!}{2!\,(m-t+2)!}\{(2^\kappa)(2^{t-2}1^{m-t+4})-\ldots\ldots\}$$
$$+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

or finally as

$$|(2^\kappa),\,(2^t)|^{m-t}+(m-t+2)|(2^\kappa),\,(2^{t-1})|^{m-t+2}$$
$$+\frac{(m-t+4)!}{2!\,(m-t+2)!}|(2^\kappa),\,(2^{t-2})|^{m-t+4}+\ldots\ldots+\frac{(m+t)!}{t!\,m!}|(2^\kappa),\,a_0|^{m+t};$$

thus the incorporation which is competent to produce every quartic form is expressible by means of incorporations of quadric forms. Generally it will be found that every form of degree n can be obtained by incorporations of quadric forms with forms of degree $n-2$.

§9.

Capitation Syzygies.

Consider the two cubic forms

$$|(2^\kappa),\,a_0|^1\equiv(2^\kappa)(1)-(2^\kappa 1),$$
$$|(2^\mu),\,a_0|^1\equiv(2^\mu)(1)-(2^\mu 1),$$

from which

$$|(2^\kappa),\,a_0|^1.|(2^\mu),\,a_0|^1=(1)^2(2^\kappa)(2^\mu)-(1)\{(2^\kappa)(2^\mu 1)+(2^\kappa 1)(2^\mu)\}+(2^\kappa 1)(2^\mu 1),$$
$$=(2)(2^\kappa)(2^\mu)+2(1^2)(2^\kappa)(2^\mu)$$
$$-(1)\{(2^\kappa)(2^\mu 1)+(2^\kappa 1)(2^\mu)\}+(2^\kappa 1)(2^\mu 1);$$

hence

$$|(2^\kappa),\ a_0|^1.|(2^\mu),\ a_0|^1-(2)(2^\kappa)(2^\mu)=(2^\kappa)|(2^\mu),\ a_0|^2+(2^\mu)|(2^\kappa),\ a_0|^2-|(2^\kappa),\ (2^\mu)|^2;$$

wherein the sinister contains sextic compounds and the dexter consists of quintic compounds, quartic compounds, and quartic, cubic, and quadric perpetuants.

We have here, therefore, a syzygy. As a simple example, put $\kappa=\mu=1$, and then

$$(3)(3)-(2)(2)(2)=(2)\{-2(2^2)\}+(2)\{-2(2^2)\}+(42)+2(3^2)+6(2^3),$$

or $$(3)^2-(2)^3=-3(42)+2(3^2)-6(2^3),$$

and this is the syzygy which yields the discriminant of the cubic. This general result arose by putting

$$(1)^2=(2)+2(1^2),$$

or we may say it came from the congruence

$$(1)^2-(2)\equiv 0 \bmod a_0.$$

Consider next the congruence

$$(1^2)^2-2(1)(1^3)-(2^2)\equiv 0 \bmod a_0;$$

proceed as follows:

$$\begin{aligned}
&|(2^\kappa),\ a_0|^4.|(2^\mu),\ a_0|^0\\
&=(1^4)(2^\kappa)(2^\mu)-(1^3)(2^\kappa 1)(2^\mu)+(1^2)(2^\kappa 1^2)(2^\mu)-(1)(2^\kappa 1^3)(2^\mu)+(2^\kappa 1^4)(2^\mu),\\
&|(2^\kappa),\ a_0|^3.|(2^\mu),\ a_0|^1\\
&=(1)(1^3)(2^\kappa)(2^\mu)-(1)(1^2)(2^\kappa 1)(2^\mu)+(1^2)(2^\kappa 1)(2^\mu 1)-(1)(2^\kappa 1^3)(2^\mu)+(2^\kappa 1^3)(2^\mu 1)\\
&\qquad-(1^3)(2^\kappa)(2^\mu 1)+(1)^2(2^\kappa 1^2)(2^\mu)-(1)(2^\kappa 1^2)(2^\mu 1),\\
&|(2^\kappa),\ a_0|^2.|(2^\mu),\ a_0|^2\\
&=(1^2)^2(2^\kappa)(2^\mu)-(1)(1^2)(2^\kappa)(2^\mu 1)+(1^2)(2^\kappa)(2^\mu 1^2)-(1)(2^\kappa 1)(2^\mu 1^2)+(2^\kappa 1^2)(2^\mu 1^2)\\
&\qquad-(1)(1^2)(2^\kappa 1)(2^\mu)+(1^2)(2^\kappa 1^2)(2^\mu)-(1)(2^\kappa 1^2)(2^\mu 1)\\
&\qquad+(1)^2(2^\kappa 1)(2^\mu 1),\\
&|(2^\kappa),\ a_0|^1.|(2^\mu),\ a_0|^3\\
&=(1)(1^3)(2^\kappa)(2^\mu)-(1)(1^2)(2^\kappa)(2^\mu 1)+(1^2)(2^\kappa 1)(2^\mu 1)-(1)(2^\kappa)(2^\mu 1^3)+(2^\kappa 1)(2^\mu 1^3)\\
&\qquad-(1^3)(2^\kappa 1)(2^\mu)+(1)^2(2^\kappa)(2^\mu 1^2)-(1)(2^\kappa 1)(2^\mu 1^2),\\
&|(2^\kappa),\ a_0|^0.|(2^\mu),\ a_0|^4\\
&=(1^4)(2^\kappa)(2^\mu)-(1^3)(2^\kappa)(2^\mu 1)+(1^2)(2^\kappa)(2^\mu 1^2)-(1)(2^\kappa)(2^\mu 1^3)+(2^\kappa)(2^\mu 1^4).
\end{aligned}$$

Consider these five identities and subtract the second and fourth from the sum of the first, third and fifth; in the result it will be seen that the second and fourth columns of terms on the right-hand side vanish identically; the third column of terms easily reduces to

$$-(2)\,|(2^\kappa),\ (2^\mu)\,|^2,$$

and the fifth column to $\qquad |(2^\kappa),\ (2^\mu)\,|^4;$

hence
$$\sum_{a=0}^{a=4}(-)^a\,|(2^\kappa),\ a_0\,|^a.\,|(2^\mu),\ a_0\,|^{4-a}$$
$$=(2^2)(2^\kappa)(2^\mu)-(2)\,|(2^\kappa),\ (2^\mu)\,|^2+|(2^\kappa),\ (2^\mu)\,|^4,$$

or finally, this is a formula representing syzygies between (4.2), (3.3) and (2.2.2) sextic compounds of perpetuants.

It is manifest that this process, here given at length, is perfectly general; the result is that the congruence $(p>1)$

$$(1^p)^2-2(1^{p-1})(1^{p+1})+2\,(1^{p-2})(1^{p+2})-\ldots+(-)^{p+1}2(1)(1^{2p-1})-(2^p)\equiv 0 \bmod a_0$$

leads to the formula of syzygies

$$\sum_{a=0}^{a=2p}(-)^{a+p}\,|(2^\kappa),\ a_0\,|^a.\,|(2^\mu),\ a_0\,|^{2p-a}$$
$$=(2^p)\,|(2^\kappa),\ (2^\mu)\,|^0-(2^{p-1})\,|(2^\kappa),\ (2^\mu)\,|^2+(2^{p-2})\,|(2^\kappa),\ (2^\mu)\,|^4-\ldots$$
$$+(-)^p\,|(2^\kappa),\ (2^\mu)\,|^{2p},$$

or, as it may be written,

$$\sum_{a=0}^{a=2p}(-)^{a+p}\,|(2^\kappa),\ a_0\,|^a.\,|(2^\mu),\ a_0\,|^{2p-a}=\sum_{\beta=0}^{\beta=p}(-)^\beta\,(2^{p-\beta})\,|(2^\kappa),\ (2^\mu)\,|^{2\beta},$$

denoting a number of complete syzygies of degree 6 and of weight $2(\varkappa+\mu+p)$.

§10.

Consider next the formation of syzygies of uneven weight. Since

$$|(2^\kappa),\ a_0\,|^2=(1^2)(2^\kappa)-(1)(2^\kappa 1)+(2^\kappa 1^2),$$
$$|(2^\mu),\ a_0\,|^1=(1)(2^\mu)-(2^\mu 1),$$
$$\therefore\ |(2^\kappa),\ a_0\,|^2.\,|(2^\mu),\ a_0\,|^1-|(2^\kappa),\ a_0\,|^1.\,|(2^\mu),\ a_0\,|^2-(1)^2\{(2^\kappa)(2^\mu 1)$$
$$-(2^\kappa 1)(2^\mu)\}\equiv 0 \bmod a_0,$$

or $\quad |(2^\kappa),\ a_0\,|^2.\,|(2^\mu),\ a_0\,|^1-|(2^\kappa),\ a_0\,|^1.\,|(2^\mu),\ a_0\,|^2-(2)\,|(2^\kappa),\ (2^\mu)\,|^1\equiv 0 \bmod a_0.$

As a verification, observe that the operation of D_6 on the sinister gives

$$|(2^{\kappa-1}),\ a_0|^1(2^{\mu-1})-(2^{\kappa-1})|(2^{\mu-1}),\ a_0|^1-|(2^{\kappa-1}),\ (2^{\mu-1})|^1,$$

which obviously vanishes identically.

Proceed now as follows:

$$\begin{aligned}&|(2^\kappa),\ a_0|^4.|(2^\mu),\ a_0|^1\\ &\quad=(1)(1^4)(2^\kappa)(2^\mu)-(1)(1^3)(2^\kappa1)(2^\mu)+(1)(1^2)(2^\kappa1^2)(2^\mu)-(1)^2(2^\kappa1^3)(2^\mu)+\ldots,\end{aligned}$$

$$\begin{aligned}&|(2^\kappa),\ a_0|^3.|(2^\mu),\ a_0|^2\\ &\quad=(1^2)(1^3)(2^\kappa)(2^\mu)-(1^2)^2(2^\kappa1)(2^\mu)+(1)(1^2)(2^\kappa1^2)(2^\mu)-(1)^2(2^\kappa1^2)(2^\mu1)+\ldots\\ &\qquad\qquad-(1)(1^3)(2^\kappa)(2^\mu1)+(1)(1^2)(2^\kappa1)(2^\mu1),\end{aligned}$$

$$\begin{aligned}&|(2^\kappa),\ a_0|^2.|(2^\mu),\ a_0|^3\\ &\quad=(1^2)(1^3)(2^\kappa)(2^\mu)-(1^2)^2(2^\kappa)(2^\mu1)+(1)(1^2)(2^\kappa)(2^\mu1^2)-(1)^2(2^\kappa1)(2^\mu1^2)+\ldots\\ &\qquad\qquad-(1)(1^3)(2^\kappa1)(2^\mu)+(1)(1^2)(2^\kappa1)(2^\mu1),\end{aligned}$$

$$\begin{aligned}&|(2^\kappa),\ a_0|^1.|(2^\mu),\ a_0|^4\\ &\quad=(1)(1^4)(2^\kappa)(2^\mu)-(1)(1^3)(2^\kappa)(2^\mu1)+(1)(1^2)(2^\kappa)(2^\mu1^2)-(1)^2(2^\kappa)(2^\mu1^3)+\ldots\end{aligned}$$

where on the right-hand side the portions omitted contain a_0 as a factor. Subtracting the second and fourth of these equations from the sum of the first and third, it will be seen that the first and third of the columns of terms on the right-hand side vanish identically, and we find

$$\begin{aligned}&\sum_{a=1}^{a=4}(-)^a|(2^\kappa),\ a_0|^a.|(2^\mu),\ a_0|^{5-a}\\ &\qquad=-\{(1^2)^2-2(1)(1^3)\}\{(2^\kappa)(2^\mu1)-(2^\kappa1)(2^\mu)\}+(1)^2|(2^\kappa),\ (2^\mu)|^3+\ldots,\end{aligned}$$

whence

$$\sum_{a=1}^{a=4}(-)^a|(2^\kappa),\ a_0|^a.|(2^\mu),\ a_0|^{5-a}+(2^2)|(2^\kappa),\ (2^\mu)|^1-(2)|(2^\kappa),\ (2^\mu)|^3\equiv 0 \bmod a_0,$$

representing another batch of sextic syzygies.

The process employed is perfectly general, and enables us to write down the syzygy of odd weight:

$$\sum_{a=1}^{a=2p}(-)^a|(2^\kappa),\ a_0|^a.|(2^\mu),\ a_0|^{2p+1-a}+\sum_{\beta=1}^{\beta=p}(-)^\beta(2^\beta)|(2^\kappa),\ (2^\mu)|^{2p+1-2\beta}\equiv 0 \bmod a_0;$$

from this syzygy, in the form of a congruence, we can at once obtain the

complete syzygy; for, operating upon the left-hand side of the congruence with D_6, we must obtain an equation, thus

$$\sum_{\alpha=1}^{\alpha=2p}(-)^{\alpha}\,|(2^{\kappa-1}),\ a_0|^{\alpha-1}.|(2^{\mu-1}),\ a_0|^{2p-\alpha}$$
$$+\sum_{\beta=1}^{\beta=p}(-)^{\beta}(2^{\beta-1})\,|(2^{\kappa-1}),\ (2^{\mu-1})|^{2p+1-2\beta}=0,$$

and herein putting

$$(\kappa,\mu,p,\alpha,\beta)=(\kappa+1,\mu+1,p+1,\alpha+1,\beta+1),$$

we obtain

$$\sum_{\alpha=0}^{\alpha=2p+1}(-)^{\alpha}\,|(2^{\kappa}),\ a_0|^{\alpha}.|(2^{\mu}),\ a_0|^{2p+1-\alpha}+\sum_{\beta=0}^{\beta=p}(-)^{\beta}(2^{\beta})\,|(2^{\kappa}),\ (2^{\mu})|^{2p+1-2\beta}=0,$$

which is the complete syzygy of degree 6 and weight $2(\kappa+\mu+p)+1$.

Finally, it may be remarked that, in this paper, the complete expression has been exhibited of a number of sextic syzygies which are enumerated by the generating function $\dfrac{x^6+x^9+x^{11}+x^{12}+x^{13}+x^{14}+x^{15}+x^{17}}{(1-x^2)(1-x^4)(1-x^6)}$; syzygies $\dfrac{x^{18}+x^{20}+x^{23}}{1-x^2.1-x^4.1-x^6}$ remain to be exhibited, but at present I do not see how to effect this.

Royal Military Academy, Woolwich, July 22d, 1887.